

New lower bounds for hypergraph Ramsey numbers

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Abstract

The *Ramsey number* $r_k(s, n)$ is the minimum N such that for every red-blue coloring of the k -tuples of $\{1, \dots, N\}$, there are s integers such that every k -tuple among them is red, or n integers such that every k -tuple among them is blue. We prove the following new lower bounds for 4-uniform hypergraph Ramsey numbers:

$$r_4(5, n) > 2^{n^{c \log n}} \quad \text{and} \quad r_4(6, n) > 2^{2^{cn^{1/5}}},$$

where c is an absolute positive constant. This substantially improves the previous best bounds of $2^{n^{c \log \log n}}$ and $2^{n^{c \log n}}$, respectively. Using previously known upper bounds, our result implies that the growth rate of $r_4(6, n)$ is double exponential in a power of n .

As a consequence, we obtain similar bounds for the k -uniform Ramsey numbers $r_k(k+1, n)$ and $r_k(k+2, n)$ where the exponent is replaced by an appropriate tower function. This almost solves the question of determining the tower growth rate for *all* classical off-diagonal hypergraph Ramsey numbers, a question first posed by Erdős and Hajnal in 1972. The only problem that remains is to prove that $r_4(5, n)$ is double exponential in a power of n .

1 Introduction

A k -uniform hypergraph H with vertex set V is a collection of k -element subsets of V . We write $K_n^{(k)}$ for the complete k -uniform hypergraph on an n -element vertex set. The *Ramsey number* $r_k(s, n)$ is the minimum N such that every red-blue coloring of the edges of $K_N^{(k)}$ contains a monochromatic red copy of $K_s^{(k)}$ or a monochromatic blue copy of $K_n^{(k)}$.

Diagonal Ramsey numbers refer to the special case when $s = n$, i.e. $r_k(n, n)$, and have been studied extensively over the past 80 years. Classic results of Erdős and Szekeres [12] and Erdős [8] imply that $2^{n/2} < r_2(n, n) \leq 2^{2n}$ for every integer $n > 2$. While small improvements have been made in both the upper and lower bounds for $r_2(n, n)$ (see [18, 4]), the constant factors in the exponents have not changed over the last 70 years.

Unfortunately for 3-uniform hypergraphs, our understanding of $r_3(n, n)$ is much less. A result of Erdős, Hajnal, and Rado [10] gives the best known lower and upper bounds for $r_3(n, n)$,

$$2^{c_1 n^2} < r_3(n, n) < 2^{2^{c_2 n}},$$

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where c_1 and c_2 are absolute constants. For $k \geq 4$, there is also a difference of one exponential between the known lower and upper bounds for $r_k(n, n)$, that is,

$$\text{twr}_{k-1}(c_1 n^2) \leq r_k(n, n) \leq \text{twr}_k(c_2 n), \quad (1)$$

where the tower function $\text{twr}_k(x)$ is defined by $\text{twr}_1(x) = x$ and $\text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}$ (see [12, 11, 9]). A notoriously difficult conjecture of Erdős, Hajnal, and Rado states that the upper bound in (1) is essentially the truth, that is, there are constructions which demonstrates that $r_k(n, n) > \text{twr}_k(cn)$, where $c = c(k)$. The crucial case is when $k = 3$, since a double exponential lower bound for $r_3(n, n)$ would verify the conjecture for all $k \geq 4$ by using the well-known stepping-up lemma of Erdős and Hajnal (see [13]).

Conjecture 1.1 (Erdős). *For $n \geq 4$ we have $r_3(n, n) > 2^{2^{cn}}$, where c is an absolute constant.*

Off-diagonal Ramsey numbers, $r_k(s, n)$, refer to the special case when k, s are fixed and n tends to infinity. It is known [1, 14, 2, 3] that $r_2(3, n) = \Theta(n^2/\log n)$, and more generally for fixed $s > 3$, $r_2(s, n) = n^{\Theta(1)}$. For 3-uniform hypergraphs, a result of Conlon, Fox, and Sudakov [6] shows that

$$2^{c_1 n \log n} \leq r_3(s, n) \leq 2^{c_2 n^{s-2} \log n},$$

where c_1 and c_2 depend only on s . For k -uniform hypergraphs, where $k \geq 4$, it is known that $r_k(s, n) \leq \text{twr}_{k-1}(n^c)$, where $c = c(s)$ [11]. By applying the Erdős-Hajnal stepping up lemma in the off-diagonal setting, it follows that

$$r_k(s, n) \geq \text{twr}_{k-1}(c'n), \quad (2)$$

for $k \geq 4$ and $s \geq 2^{k-1} - k + 3$, where $c' = c'(s)$. In 1972, Erdős and Hajnal [9] conjectured that (2) holds for every fixed $k \geq 4$ and $s \geq k + 1$. Actually, this was part of a more general conjecture that they posed in that paper (see [16, 17] for details). In [5], Conlon, Fox, and Sudakov verified the Erdős-Hajnal conjecture for all $s \geq \lceil 5k/2 \rceil - 3$. Very recently, the current authors [16] and independently Conlon, Fox, and Sudakov [7] verified the conjecture for all $s \geq k + 3$ (using different constructions). Since $2^{k-1} - k + 3 = \lceil 5k/2 \rceil - 3 = k + 3 = 7$ when $k = 4$, all three of these approaches succeed in proving a double exponential lower bound for $r_4(7, n)$ but fail for $r_4(6, n)$ and $r_4(5, n)$. Just as for diagonal Ramsey numbers, a double exponential in n^c lower bound for $r_4(5, n)$ and $r_4(6, n)$ would imply $r_k(k + 1, n) > \text{twr}_{k-1}(n^{c'})$ and $r_k(k + 2, n) > \text{twr}_{k-1}(n^{c'})$ respectively, for all fixed $k \geq 5$. This follows from a variant of the stepping-up lemma that we will describe in Section 2. Therefore, the difficulty in verifying (2) for the two remaining cases, $s = k + 1$ and $k + 2$, is due to our lack of understanding of $r_4(5, n)$ and $r_4(6, n)$. Consequently, showing double exponential lower bounds for $r_4(5, n)$ and $r_4(6, n)$ are the only two problems that remain to determine the tower growth rate for all off-diagonal hypergraph Ramsey numbers.

Until very recently, the only lower bound for both $r_4(5, n)$ and $r_4(6, n)$ was 2^{cn} , which was implicit in the paper of Erdős and Hajnal [9]. Our results in [15, 16] improved both these bounds to

$$r_4(5, n) > 2^{n^{c \log \log n}} \quad \text{and} \quad r_4(6, n) > 2^{n^{c \log n}} \quad (3)$$

and these are the current best known bounds. As mentioned above, the bounds in (3) imply the corresponding improvements to the lower bounds for $r_k(k + 1, n)$ and $r_k(k + 2, n)$. In this paper we further substantially improve both lower bounds in (3).

Theorem 1.2. *For all $n \geq 6$,*

$$r_4(5, n) > 2^{n^{c \log n}} \quad \text{and} \quad r_4(6, n) > 2^{2^{cn^{1/5}}},$$

where $c > 0$ is an absolute constant.

Using the stepping-up lemma (see Section 2) we obtain the following.

Corollary 1.3. *For $n > k \geq 5$, there is a $c = c(k) > 0$ such that*

$$r_k(k+1, n) > \text{twr}_{k-2}(n^{c \log n}) \quad \text{and} \quad r_k(k+2, n) > \text{twr}_{k-1}(cn^{1/5}).$$

A standard argument in Ramsey theory together with results in [6] for 3-uniform hypergraph Ramsey numbers yields the upper bound $r_k(k+2, n) < \text{twr}_{k-1}(c'n^3 \log n)$ so we now know the tower growth rate of $r_k(k+2, n)$.

In [16], we established a connection between diagonal and off-diagonal Ramsey numbers. In particular, we showed that a solution to Conjecture 1.1 implies a solution to the following conjecture.

Conjecture 1.4. *For $n \geq 5$, there is an absolute constant $c > 0$ such that $r_4(5, n) > 2^{2^{n^c}}$.*

Our approach is inspired by the stepping-up method but there are several new ideas that are employed. One major idea is to apply stepping-up starting from a graph to construct a 4-uniform hypergraph, rather than the usual method of going from a 3-uniform hypergraph to a 4-uniform hypergraph. Although this approach was implicitly developed in [16], here we use it explicitly.

For more related Ramsey-type results for hypergraphs, we refer the interested reader to [16, 15, 17]. All logarithms are in base 2 unless otherwise stated. For the sake of clarity of presentation, we omit floor and ceiling signs whenever they are not crucial.

2 The stepping-up lemma and proof of Lemma 2.1

The proof of our main result, Theorem 1.2, follows by applying a variant of the classic Erdős-Hajnal stepping-up lemma. In this section, we describe the stepping-up procedure and sketch the proof of Lemma 2.1 below which is used to prove Corollary 1.3. The particular case below can be found in [15], though a special case of Lemma 2.1 was communicated to us independently by Conlon, Fox, and Sudakov [7].

Lemma 2.1. *For $k \geq 5$ and $n \geq s \geq k+1$, we have $r_k(s, 2kn) > 2^{r_{k-1}(s-1, n)-1}$.*

Proof. Let $k \geq 5$, $n \geq s \geq k+1$, and set $A = \{0, 1, \dots, N-1\}$ where $N = r_{k-1}(s-1, n) - 1$. Let $\phi : \binom{A}{k-1} \rightarrow \{\text{red}, \text{blue}\}$ be a red/blue coloring of the $(k-1)$ -tuples of A such that there is no monochromatic red copy of $K_{s-1}^{(k-1)}$ and no monochromatic blue copy of $K_n^{(k-1)}$. We know ϕ exists by definition of N . Set $V = \{0, 1, \dots, 2^N - 1\}$. In what follows, we will use ϕ to define a red/blue coloring $\chi : \binom{V}{k} \rightarrow \{\text{red}, \text{blue}\}$ of the k -tuples of V , such that χ does not produce a monochromatic

red copy of $K_s^{(k)}$, and does not produce a monochromatic blue copy of $K_{2kn}^{(k)}$. First, let us observe several properties of V .

For any $v \in V$, write $v = \sum_{i=0}^{N-1} v(i)2^i$ with $v(i) \in \{0, 1\}$ for each i . For $u \neq v$, let $\delta(u, v) \in A$ denote the largest i for which $u(i) \neq v(i)$. Notice that we have the following stepping-up properties (see [13])

Property I: For every triple $u < v < w$, $\delta(u, v) \neq \delta(v, w)$.

Property II: For $v_1 < \dots < v_r$, $\delta(v_1, v_r) = \max_{1 \leq j \leq r-1} \delta(v_j, v_{j+1})$.

Property III: For every 4-tuple $v_1 < \dots < v_4$, if $\delta(v_1, v_2) > \delta(v_2, v_3)$, then $\delta(v_1, v_2) \neq \delta(v_3, v_4)$.

Note that if $\delta(v_1, v_2) < \delta(v_2, v_3)$, it is possible that $\delta(v_1, v_2) = \delta(v_3, v_4)$.

Property IV: For $v_1 < \dots < v_r$, set $\delta_j = \delta(v_j, v_{j+1})$ and suppose that $\delta_1, \dots, \delta_{r-1}$ forms a monotone sequence. Then for every subset of k -vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$, where $v_{i_1} < \dots < v_{i_k}$, $\delta(v_{i_1}, v_{i_2}), \delta(v_{i_2}, v_{i_3}), \dots, \delta(v_{i_{k-1}}, v_{i_k})$ forms a monotone sequence. Moreover, for every subset of $k-1$ vertices $\delta_{j_1}, \delta_{j_2}, \dots, \delta_{j_{k-1}}$, there are k vertices v_{i_1}, \dots, v_{i_k} such that $\delta(v_{i_t}, v_{i_{t+1}}) = \delta_{j_t}$.

Given any k -tuple $v_1 < v_2 < \dots < v_k$ of V , consider the integers $\delta_i = \delta(v_i, v_{i+1})$, $1 \leq i \leq k-1$. We say that δ_i is a *local minimum* if $\delta_{i-1} > \delta_i < \delta_{i+1}$, a *local maximum* if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and a *local extremum* if it is either a local minimum or a local maximum. Since $\delta_{i-1} \neq \delta_i$ for every i , every nonmonotone sequence $\delta_1, \dots, \delta_{k-1}$ has a local extremum.

Using $\phi : \binom{A}{k-1} \rightarrow \{\text{red}, \text{blue}\}$, we define $\chi : \binom{V}{k} \rightarrow \{\text{red}, \text{blue}\}$ as follows. For $v_1 < \dots < v_k$ and $\delta_i = \delta(v_i, v_{i+1})$, we define $\chi(v_1, \dots, v_k) = \text{red}$ if

- (a) $\delta_1, \dots, \delta_{k-1}$ forms a monotone sequence and $\phi(\delta_1, \dots, \delta_{k-1}) = \text{red}$, or if
- (b) $\delta_1, \dots, \delta_{k-1}$ forms a *zig-zag* sequence such that δ_2 is a local maximum. In other words, $\delta_1 < \delta_2 > \delta_3 < \delta_4 > \dots$.

Otherwise $\chi(v_1, \dots, v_k) = \text{blue}$.

For sake of contradiction, suppose χ produces a monochromatic red copy of $K_s^{(k)}$ on vertices $v_1 < \dots < v_s$, and let $\delta_i = \delta(v_i, v_{i+1})$. If $\delta_1, \delta_2, \dots, \delta_{s-1}$ forms monotone sequence, then by Property IV, ϕ colors every $(k-1)$ -tuple in the set $\{\delta_1, \dots, \delta_{s-1}\}$ red, which is a contradiction. Let δ_i denote the first local extremum in the sequence $\delta_1, \dots, \delta_{s-1}$. It is easy to see that δ_i is a local maximum since otherwise we would get a contradiction. Suppose $i+k-1 \leq s$. If δ_{i+1} is not a local extremum, then $\chi(v_{i-1}, v_i, v_{i+1}, \dots, v_{i+k-2}) = \text{blue}$ which is a contradiction. If δ_{i+1} is a local extremum, then it must be a local minimum which implies that $\chi(v_i, v_{i+1}, \dots, v_{i+k-1}) = \text{blue}$, contradiction. Therefore we can assume that $i+k-1 > s$, which implies $i \geq 3$ since $s \geq k+1$. However, this implies that either $\chi(v_{i-2}, v_{i-1}, \dots, v_{i+k-3}) = \text{blue}$ or $\chi(v_{s-k+1}, v_{s-k+2}, \dots, v_s) = \text{blue}$, contradiction. Hence, χ does not produce a monochromatic red copy of $K_s^{(k)}$ in V .

Let $m = 2kn$. For sake of contradiction, suppose χ produces a monochromatic blue copy of $K_m^{(k)}$ on vertices v_1, \dots, v_m and let $\delta_i = \delta(v_i, v_{i+1})$. By Property IV, there is no x such that

$\delta_x, \delta_{x+1}, \dots, \delta_{x+n-1}$ forms a monotone sequence. Indeed, otherwise ϕ would produce a monochromatic blue copy of $K_n^{(k-1)}$ on vertices $\delta_x, \delta_{x+1}, \dots, \delta_{x+n-1}$. Therefore, we can set $\delta_{i_1}, \dots, \delta_{i_k}$ to be the first k local minimums in the sequence $\delta_1, \dots, \delta_{m-1}$. However, by Property II, χ colors the first k vertices in the set $\{v_{i_1}, v_{i_1+1}, v_{i_2}, v_{i_2+1}, \dots, v_{i_k}, v_{i_k+1}\}$ red which is a contradiction. This completes the proof of Lemma 2.1. \square

3 A double exponential lower bound for $r_4(6, n)$

The lower bound for $r_4(6, n)$ follows by applying a variant the Erdős-Hajnal stepping up lemma. We start with the following simple lemma.

Lemma 3.1. *There is an absolute constant $c > 0$ such that the following holds. For every $n \geq 6$, there is a red/blue coloring ϕ of the pairs of $\{0, 1, \dots, \lfloor 2^{cn} \rfloor - 1\}$ such that*

1. *there are no two disjoint n -sets $A, B \subset \{0, 1, \dots, \lfloor 2^{cn} \rfloor - 1\}$, such that $\phi(a, b) = \text{red}$ for every $a \in A$ and $b \in B$, or $\phi(a, b) = \text{blue}$ for every $a \in A$ and $b \in B$ (i.e., no monochromatic $K_{n,n}$),*
2. *there is no n -set $A \subset \{0, 1, \dots, \lfloor 2^{cn} \rfloor - 1\}$ such that every triple $a_i, a_j, a_k \in A$, where $a_i < a_j < a_k$, avoids the pattern $\phi(a_i, a_j) = \phi(a_j, a_k) = \text{blue}$ and $\phi(a_i, a_k) = \text{red}$.*

Proof. Set $N = \lfloor 2^{cn} \rfloor$, where c is a sufficiently small constant that will be determined later. Consider the red/blue coloring ϕ of the pairs (edges) of $\{0, 1, \dots, N - 1\}$, where each edge has probability $1/2$ of being a particular color independent of all other edges. Then the expected number of monochromatic copies of the complete bipartite graph $K_{n,n}$ is at most

$$\binom{N}{n}^2 2^{-n^2+1} < 1/3,$$

for c sufficiently small and $n \geq 6$.

We call a triple $a_i, a_j, a_k \in \{0, 1, \dots, N - 1\}$ *bad* if $a_i < a_j < a_k$ and $\phi(a_i, a_j) = \phi(a_j, a_k) = \text{blue}$ and $\phi(a_i, a_k) = \text{red}$. Otherwise, we call the triple (a_i, a_j, a_k) *good*. Now, let us estimate the expected number of sets $A \subset \{0, 1, \dots, N - 1\}$ of size n such that every triple in A is good. For a given triple $a_i, a_j, a_k \in \{0, 1, \dots, N - 1\}$, where $a_i < a_j < a_k$, the probability that (a_i, a_j, a_k) is good is $7/8$. Let $A = \{a_1, \dots, a_n\}$ be a set of n vertices in $\{0, 1, \dots, N - 1\}$, where $a_1 < \dots < a_n$. Let S be a partial Steiner $(n, 3, 2)$ -system with vertex set A , that is, S is a 3-uniform hypergraph such that each 2-element set of vertices is contained in at most one edge in S . Moreover, S satisfies $|S| = c'n^2$. It is known that such a system exists. Then the probability that every triple in A is good is at most the probability that every edge in S is good. Since the edges in S are independent, that is no two edges have more than one vertex in common, the probability that every triple in A is good is at most $(\frac{7}{8})^{|S|} \leq (\frac{7}{8})^{c'n^2}$. Therefore, the expected number of sets of size n with every triple being good is at most

$$\binom{N}{n} \left(\frac{7}{8}\right)^{c'n^2} < 1/3,$$

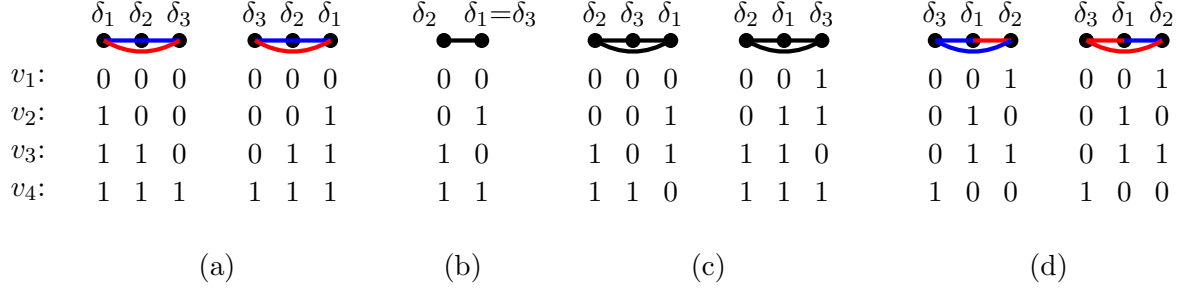


Figure 1: Examples of $v_1 < v_2 < v_3 < v_4$ and $\delta_1 = \delta(v_1, v_2), \delta_2 = \delta(v_2, v_3), \delta_3 = \delta(v_3, v_4)$, such that $\chi(v_1, v_2, v_3, v_4) = \text{red}$. For each case, v_i is represented in binary form with the left-most entry being the most significant bit.

for an appropriate choice for c . By Markov's inequality and the union bound, we can conclude that there is a coloring ϕ with the desired properties. \square

Let $c > 0$ be the constant from the lemma above, and let $A = \{0, 1, \dots, \lfloor 2^{cn} \rfloor - 1\}$ and $\phi : \binom{A}{2} \rightarrow \{\text{red}, \text{blue}\}$ be a 2-coloring of the pairs of A with the properties described above. Let $V = \{0, 1, \dots, N - 1\}$, where $N = \lfloor 2^{2^{cn}} \rfloor$. In what follows, we will use ϕ to define a red/blue coloring $\chi : \binom{V}{4} \rightarrow \{\text{red}, \text{blue}\}$ of the 4-tuples of V such that χ does not produce a monochromatic red copy of $K_6^{(4)}$ and does not produce a monochromatic blue copy of $K_{32n^5}^{(4)}$. This would imply the desired lower bound for $r_4(6, n)$. For $v_1 < v_2 < v_3 < v_4$ and $\delta_i = \delta(v_i, v_{i+1})$, we set $\chi(v_1, v_2, v_3, v_4) = \text{red}$ if

- (a) $\delta_1, \delta_2, \delta_3$ forms a monotone sequence and the triple $(\delta_1, \delta_2, \delta_3)$ is *bad*, that is, $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$, or
- (b) $\delta_1 < \delta_2 > \delta_3$ and $\delta_1 = \delta_3$, or
- (c) $\delta_1 < \delta_2 > \delta_3$, $\delta_1 \neq \delta_3$, and the set $\{\delta_1, \delta_2, \delta_3\}$ is monochromatic with respect to ϕ , or
- (d) $\delta_1 > \delta_2 < \delta_3$, $\delta_1 < \delta_3$, and $\phi(\delta_3, \delta_1) = \phi(\delta_3, \delta_2)$ and $\phi(\delta_1, \delta_2) \neq \phi(\delta_3, \delta_1)$.

See Figure 1 for small examples. Otherwise, $\chi(v_1, v_2, v_3, v_4) = \text{blue}$.

For sake of contradiction, suppose that the coloring χ produces a red $K_6^{(4)}$ on vertices $v_1 < \dots < v_6$, and let $\delta_i = \delta(v_i, v_{i+1})$, $1 \leq i \leq 5$. Let us first consider the following cases for $\delta_1, \dots, \delta_4$, which corresponds to the vertices, v_1, \dots, v_5 .

Case 1. Suppose that $\delta_1, \dots, \delta_4$ forms a monotone sequence. If $\delta_1 > \dots > \delta_4$, then we have $\phi(\delta_1, \delta_3) = \text{red}$ since $\chi(v_1, v_2, v_3, v_4) = \text{red}$. However, this implies that $\chi(v_1, v_3, v_4, v_5) = \text{blue}$ since $\delta(v_1, v_3) = \delta_1$ by Property II, contradiction. A similar argument follows if $\delta_1 < \dots < \delta_4$.

Case 2. Suppose $\delta_1 > \delta_2 > \delta_3 < \delta_4$. By Property III, $\delta_4 \neq \delta_2, \delta_1$. Since $\delta_1 > \delta_2 > \delta_3$, this implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$. Since $\delta(v_2, v_4) = \delta_2$ and $\chi(v_1, v_2, v_4, v_5) =$

red, we have $\delta_4 > \delta_1$. Hence $\phi(\delta_4, \delta_3) = \phi(\delta_4, \delta_2) = \text{red}$. However, since $\delta(v_1, v_3) = \delta_1$ by Property II, we have $\chi(v_1, v_3, v_4, v_5)$ is blue, contradiction.

Case 3. Suppose $\delta_1 < \delta_2 < \delta_3 > \delta_4$. This implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$. Suppose $\delta_4 = \delta_2$. Since $\delta(v_1, v_3) = \delta_2$ and $\delta_2 < \delta_3 > \delta_4$, this implies the triple $(\delta_1, \delta_2, \delta_4)$ forms a monochromatic blue set with respect to ϕ , which is a contradiction. A similar argument follows in the case that $\delta_4 = \delta_1$. So we can assume $\delta_4 \neq \delta_1, \delta_2$. Since $\chi(v_2, v_3, v_4, v_5) = \text{red}$, the triple $\{\delta_2, \delta_3, \delta_4\}$ forms a monochromatic blue set with respect to ϕ . By Property II we have $\delta(v_2, v_4) = \delta_3$ and $\delta_1 < \delta(v_2, v_4) > \delta_4$. This implies that $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction.

Case 4. Suppose $\delta_1 < \delta_2 > \delta_3 > \delta_4$. This implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) = \text{blue}$ and $\phi(\delta_2, \delta_4) = \text{red}$. Suppose $\delta_1 = \delta_3$. By Property II, we have $\delta(v_2, v_4) = \delta_2$. However, $\chi(v_1, v_2, v_4, v_5) = \text{red}$ and $\delta_1 < \delta_2 > \delta_4$ implies that the triple $(\delta_1, \delta_2, \delta_4)$ must form a monochromatic set with respect to ϕ , contradiction. A similar argument follows if $\delta_1 = \delta_4$. Therefore, we can assume that $\delta_1 \neq \delta_3, \delta_4$. Since $\chi(v_1, v_2, v_3, v_4) = \text{red}$, the triple $(\delta_1, \delta_2, \delta_3)$ forms a monochromatic blue set with respect to ϕ . By Property II we have $\delta(v_2, v_4) = \delta_2$ and $\delta_1 < \delta(v_2, v_4) > \delta_4$. This implies $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction.

Case 5. Suppose $\delta_1 > \delta_2 < \delta_3 < \delta_4$. Note that by Property III, $\delta_1 \neq \delta_3, \delta_4$. Since $\delta_1, \delta_2, \delta_3$ forms a monotone sequence, this implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) = \text{blue}$ and $\phi(\delta_2, \delta_4) = \text{red}$. Moreover, we must have $\delta_1 < \delta_3$ since $\chi(v_1, v_2, v_3, v_4) = \text{red}$. Hence $\phi(\delta_3, \delta_1) = \text{blue}$ and $\phi(\delta_1, \delta_2) = \text{red}$. However, since $\delta(v_3, v_5) = \delta_4$, we have $\delta_1 > \delta_2 < \delta(v_3, v_5)$ and $\chi(v_1, v_2, v_3, v_5) = \text{blue}$, contradiction.

Case 6. Suppose $\delta_1 < \delta_2 > \delta_3 < \delta_4$. Then we must also have $\delta_4 > \delta_2$ since $\chi(v_2, v_3, v_4, v_5) = \text{red}$. By Property II, $\delta(v_3, v_5) = \delta_4$ and we have $\delta_1 < \delta_2 < \delta(v_3, v_5)$. Since $\chi(v_1, v_2, v_3, v_5) = \text{red}$, we have $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_4) = \text{blue}$ and $\phi(\delta_1, \delta_4) = \text{red}$. Since $\chi(v_1, v_2, v_3, v_4) = \text{red}$, the triple $(\delta_1, \delta_2, \delta_3)$ forms a monochromatic blue set. However, this implies that $\chi(v_2, v_3, v_4, v_5) = \text{blue}$, contradiction.

Now if v_1, \dots, v_5 and $\delta_1, \dots, \delta_4$ does not fall into one of the 6 cases above, then we must have $\delta_1 > \delta_2 < \delta_3 > \delta_4$. However, this implies that v_2, \dots, v_6 and $\delta_2, \dots, \delta_5$ does fall into one of the 6 cases above, which implies our contradiction. Therefore, χ does not produce a monochromatic red copy of $K_6^{(4)}$ in our 4-uniform hypergraph.

Next we show that there is no blue $K_m^{(4)}$ in coloring χ , where $m = 32n^5$. For sake of contradiction, suppose we have vertices $v_1, \dots, v_m \in V$ such that $v_1 < \dots < v_m$, and χ colors every 4-tuple in the set $\{v_1, \dots, v_m\}$ blue. Let $\delta_i = \delta(v_i, v_{i+1})$ for $1 \leq i \leq m-1$.

Set $\delta_1^* = \max\{\delta_1, \dots, \delta_m\}$, where $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$. Set

$$V_1 = \{v_1, v_2, \dots, v_{i_1}\} \quad \text{and} \quad V_2 = \{v_{i_1+1}, v_{i_1+2}, \dots, v_m\}.$$

Now we establish the following lemma.

Lemma 3.2. *We have either $|V_1| < m/2n$ or $|V_2| < m/2n$.*

Before we prove Lemma 3.2, let us finish the argument that χ does not color every 4-tuple in the set $\{v_1, \dots, v_m\}$ blue via the following lemma which will also be used later in the paper.

Lemma 3.3. *If Lemma 3.2 holds, then χ colors a 4-tuple in the set $\{v_1, \dots, v_m\}$ red.*

Proof. We greedily construct a set $D_t = \{\delta_1^*, \delta_2^*, \dots, \delta_t^*\} \subset \{\delta_1, \delta_2, \dots, \delta_m\}$ and a set $S_t \subset \{v_1, \dots, v_m\}$ such that the following holds.

1. We have $\delta_1^* > \dots > \delta_t^*$, where $\delta_j^* = \delta(v_{i_j}, v_{i_j+1})$.
2. The indices of the vertices in S_t are consecutive, that is, $S_t = \{v_r, v_{r+1}, \dots, v_{s-1}, v_s\}$ for $1 \leq r < s \leq n$. Moreover, $\delta_t^* > \max\{\delta_r, \delta_{r+1}, \dots, \delta_{s-1}\}$.
3. $|S_t| > m - tm/2n$.
4. For each $\delta_j^* = \delta(v_{i_j}, v_{i_j+1}) \in D_t$, consider the set of vertices

$$S = \{v_{i_j+1}, v_{i_j+1+1}, v_{i_j+2}, v_{i_j+2+1} \dots, v_{i_t}, v_{i_t+1}\} \cup S_t.$$

Then either every element in S is greater than v_{i_j} or every element in S is less than v_{i_j+1} . In the former case we will label δ_j^* *white*, in the latter case we label it *black*.

We start with the $D_0 = \emptyset$ and $S_0 = \{v_1, \dots, v_m\}$. Having obtained $D_{t-1} = \{\delta_1^*, \dots, \delta_{t-1}^*\}$ and $S_{t-1} = \{v_r, \dots, v_s\}$, where $1 \leq r < s \leq n$, we construct D_t and S_t as follows. Let $\delta_t^* = \delta(v_{i_t}, v_{i_t+1})$ be the unique largest element in $\{\delta_r, \delta_{r+1}, \dots, \delta_{s-1}\}$, and set $D_t = D_{t-1} \cup \delta_t^*$. The uniqueness of δ_t^* follows from Properties I and II. We partition $S_{t-1} = T_1 \cup T_2$, where $T_1 = \{v_r, v_{r+1}, \dots, v_{i_t}\}$ and $T_2 = \{v_{i_t+1}, v_{i_t+2}, \dots, v_s\}$. By Lemma 3.2, either $|T_1| < m/2n$ or $|T_2| < m/2n$. If $|T_1| < m/2n$, we set $S_t = T_2$ and label δ_t^* white. Likewise, if $|T_2| < m/2n$, we set $S_t = T_1$ and label δ_t^* black. By induction, we have

$$|S_t| > |S_{t-1}| - m/2n \geq (m - (t-1)m/2n) - m/2n = m - tm/2n.$$

Since $|S_0| = m$ and $|S_t| \geq 1$ for $t = 2n$, we can construct $D_{2n} = \{\delta_1^*, \dots, \delta_{2n}^*\}$ with the desired properties. By the pigeonhole principle, there are at least n elements in D_{2n} with the same label, say *white*. The other case will follow by a symmetric argument. We remove all black labeled elements in D_{2n} , and let $\{\delta_{j_1}^*, \dots, \delta_{j_n}^*\}$ be the resulting set.

Now consider the vertices $v_{j_1}, v_{j_2}, \dots, v_{j_n}, v_{j_n+1} \in V$. By construction and by Property II, we have $v_{j_1} < v_{j_2} < \dots < v_{j_n} < v_{j_n+1}$ and $\delta(v_{j_1}, v_{j_2}) = \delta_{i_{j_1}}^*, \delta(v_{j_2}, v_{j_3}) = \delta_{i_{j_2}}^*, \dots, \delta(v_{j_n}, v_{j_n+1}) = \delta_{i_{j_n}}^*$. Therefore, we have a monotone sequence

$$\delta(v_{j_1}, v_{j_2}) > \delta(v_{j_2}, v_{j_3}) > \dots > \delta(v_{j_n}, v_{j_n+1}).$$

By Lemma 3.1, there is a bad triple in the set $\{\delta_{j_1}^*, \dots, \delta_{j_n}^*\}$ with respect to ϕ . By Property IV, χ does not color every 4-tuple in $V = \{v_1, \dots, v_m\}$ blue, which completes the proof of Lemma 3.3. \square

Now let us go back and prove Lemma 3.2. First, we make the following observation.

Observation 3.4. *Let $v_1 < \dots < v_m \in V$ such that χ colors every 4-tuple in the set $\{v_1, \dots, v_m\}$ blue. Then for $\delta_i = \delta(v_i, v_{i+1})$, $\delta_i \neq \delta_j$ for $1 \leq i < j < m$.*

Proof. For sake of contradiction, suppose $\delta_i = \delta_j$ for $i \neq j$. By Property I, $j \neq i + 1$. Without loss of generality, we can assume that for all r such that $i < r < j$, $\delta_r \neq \delta_i$. Set $\delta_r = \max\{\delta_{i+1}, \delta_{i+2}, \dots, \delta_{j-1}\}$, and notice that $\delta(v_{i+1}, v_j) = \delta_r$ by Property II. Now if $\delta_r > \delta_i = \delta_j$, then $\chi(v_i, v_{i+1}, v_j, v_{j+1}) = \text{red}$ and we have a contradiction. If $\delta_r < \delta_i$, then this would contradict Property III. Hence, the statement follows. \square

Proof of Lemma 3.2. For sake of contradiction, suppose $|V_1|, |V_2| \geq m/2n = 16n^4$. Recall that $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$, $V_1 = \{v_1, v_2, \dots, v_{i_1}\}$, $V_2 = \{v_{i_1+1}, v_{i_1+1}, \dots, v_m\}$, and set $A_1 = \{\delta_1, \dots, \delta_{i_1-1}\}$ and $A_2 = \{\delta_{i_1+1}, \dots, \delta_{m-1}\}$. For $i \in \{1, 2\}$, let us partition $A_i = A_i^r \cup A_i^b$ where

$$A_i^r = \{\delta_j \in A_i : \phi(\delta_1^*, \delta_j) = \text{red}\} \quad \text{and} \quad A_i^b = \{\delta_j \in A_i : \phi(\delta_1^*, \delta_j) = \text{blue}\}.$$

By the pigeonhole principle, either $|A_2^b| \geq 8n^4$ or $|A_2^r| \geq 8n^4$. Without loss of generality, we can assume that $|A_2^b| \geq 8n^4$ since a symmetric argument would follow otherwise.

Fix $\delta_{j_1} \in A_1^b$ and $\delta_{j_2} \in A_2^b$, and recall that $\delta_{j_1} = \delta(v_{j_1}, v_{j_1+1})$ and $\delta_{j_2} = \delta(v_{j_2}, v_{j_2+1})$. By Observation 3.4, $\delta_{j_1} \neq \delta_{j_2}$, and by Property II, we have $\delta(v_{j_1+1}, v_{j_2}) = \delta_1^*$. Since $\chi(v_{j_1}, v_{j_1+1}, v_{j_2}, v_{j_2+1}) = \text{blue}$, this implies that $\phi(\delta_{j_1}, \delta_{j_2}) = \text{red}$. By Lemma 3.1 and Observation 3.4, we have $|A_1^b| < n$. Indeed, otherwise we would have a monochromatic red copy of $K_{n,n}$ in A with respect to ϕ . Therefore we have $|A_1^r| \geq 16n^4 - n - 1$. Again by the pigeonhole principle, there is a subset $B \subset A_1^r$ of size at least $(16n^4 - n - 1)/n \geq 8n^3 - 1$, such that $B = \{\delta_j, \delta_{j+1}, \dots, \delta_{j+8n^3-2}\}$, and whose corresponding vertices are $U = \{v_j, v_{j+1}, \dots, v_{j+8n^3-1}\}$. For simplicity and without loss of generality, let us rename $U = \{u_1, \dots, u_{8n^3}\}$ and $\delta_i = \delta(u_i, u_{i+1})$ for $1 \leq i \leq 8n^3 - 1$.

Just as before, we greedily construct a set $D_t = \{\delta_1^*, \dots, \delta_t^*\} \subset \delta_1^* \cup \{\delta_1, \dots, \delta_{8n^3-1}\}$ and a set $S_t \subset \{u_1, \dots, u_{8n^3}\}$ such that the following holds.

1. We have $\delta_1^* > \dots > \delta_t^*$, where $\delta_j^* = \delta(u_{i_j}, u_{i_j+1})$ for $i \geq 2$.
2. For each $\delta_j^* = \delta(u_{i_j}, u_{i_j+1}) \in D_t$, consider the set of vertices

$$S = \{u_{i_{j+1}}, u_{i_{j+1}+1}, \dots, u_{i_h}, u_{i_h+1}\} \cup S_t.$$

Then either every element in S is greater than u_{i_j} or every element in S is less than u_{i_j+1} . In the former case we will label $\delta_{i_j}^*$ *white*, in the latter case we label it *black*.

3. The indices of the vertices in S_t are consecutive, that is, $S_t = \{u_r, u_{r+1}, \dots, u_{s-1}, u_s\}$ for $1 \leq r < s \leq n$. Set $B_t = \{\delta_r, \delta_{r+1}, \dots, \delta_{u_s-1}\}$.
4. for each $\delta_j^* \in D_t$, either $\phi(\delta_j^*, \delta) = \text{red}$ for every $\delta \in \{\delta_{j+1}^*, \delta_{j+2}^*, \dots, \delta_t^*\} \cup B_t$, or $\phi(\delta_j^*, \delta) = \text{blue}$ for every $\delta \in \{\delta_{j+1}^*, \delta_{j+2}^*, \dots, \delta_t^*\} \cup B_t$.
5. We have $|S_t| \geq 8n^3 - (t-1)2n^2$.

We start with $S_1 = U = \{u_1, \dots, u_{8n^3}\}$ and $D_1 = \{\delta_1^*\}$, where we recall that $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$. Having obtained $D_{t-1} = \{\delta_1^*, \dots, \delta_{t-1}^*\}$ and $S_{t-1} = \{u_r, \dots, u_s\}$, $1 \leq r < s \leq n$, we construct D_t and S_t as follows. Let $\delta_t^* = \delta(u_{i_t}, u_{i_t+1})$ be the unique largest element in $\{\delta_r, \delta_{r+1}, \dots, \delta_{s-1}\}$, and set $D_t = D_{t-1} \cup \delta_t^*$. The uniqueness of δ_t^* follows from Properties I and II. Let us partition $S_t = T_1 \cup T_2$, where $T_1 = \{u_r, u_{r+1}, \dots, u_{i_t}\}$ and $T_2 = \{u_{i_t+1}, u_{i_t+1}+2, \dots, u_s\}$. Now we make the following observation.

Observation 3.5. *We have $|T_1| < 2n^2$ or $|T_2| < 2n^2$.*

Proof. For sake of contradiction, suppose $|T_1|, |T_2| \geq 2n^2$ and let $B_1 = \{\delta_r, \delta_{r+1}, \dots, \delta_{i_t-1}\}$ and $B_2 = \{\delta_{i_1+1}, \delta_{i_t+2}, \dots, \delta_{s-1}\}$. Notice that for every $\delta \in B_2$ we have $\phi(\delta_t^*, \delta) = \text{red}$. Indeed, suppose for $\delta = \delta(u_\ell, u_{\ell+1}) \in B_2$ we have $\phi(\delta_t^*, \delta) = \text{blue}$. Recall $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$, $\delta_t^* = \delta(u_{i_t}, u_{i_t+1})$, where

$$u_{i_t} < u_{i_t+1} < u_\ell < u_{\ell+1} < v_{i_1} < v_{i_1+1}.$$

Consider the vertices $v_{i_1+1}, u_{i_t}, u_\ell, u_{\ell+1}$. By definition of χ , we have $\chi(u_{i_t}, u_\ell, u_{\ell+1}, v_{i_1+1}) = \text{red}$, contradiction. Therefore, by the same argument as above, there are less than n elements $\delta \in B_1$ such that $\phi(\delta_t^*, \delta) = \text{red}$. Since $|T_1| > 2n^2$, by the pigeonhole principle, there is a set of $n+1$ consecutive vertices $\{u_\ell, u_{\ell+1}, \dots, u_{\ell+n}\} \subset T_1$ and the subset $\{\delta_\ell, \delta_{\ell+1}, \dots, \delta_{\ell+n-1}\} \subset B_1$ such that $\phi(\delta_t^*, \delta) = \text{blue}$ for every $\delta \in \{\delta_\ell, \delta_{\ell+1}, \dots, \delta_{\ell+n-1}\}$. Notice that

$$\delta_\ell < \delta_{\ell+1} < \dots < \delta_{\ell+n-1}.$$

Indeed, suppose $\delta_r > \delta_{r+1}$ for some $r \in \{\ell, \ell+1, \dots, \ell+n-2\}$. Then $\phi(\delta_r, \delta_{r+1}) = \text{red}$ implies that $\chi(u_{i_t+1}, u_r, u_{r+1}, u_{r+2}) = \text{red}$, contradiction. Likewise if $\phi(\delta_r, \delta_{r+1}) = \text{blue}$, then $\chi(v_{i_1+1}, u_r, u_{r+1}, u_{r+2}) = \text{red}$, contradiction. However, by Lemma 3.1, there is a bad triple in $\{\delta_\ell, \delta_{\ell+1}, \dots, \delta_{\ell+n-1}\}$ with respect to ϕ . Since $\delta_\ell, \delta_{\ell+1}, \dots, \delta_{\ell+n-1}$ forms a monotone sequence, by Property IV, χ colors some 4-tuple in the set $\{u_\ell, u_{\ell+1}, \dots, u_{\ell+n}\}$ red, contradiction. Hence the statement follows. \square

If $|T_1| < 2n^2$, we set $S_t = T_2$. Otherwise by Observation 3.5 we have $|T_2| < 2n^2$ and we set $S_t = T_1$. Hence $|S_t| > |S_{t-1}| - 2n^2$.

Since $|S_1| = |U| = 8n^3$, we have $|S_t| > 0$ for $t = 2n$. Therefore, we can construct $D_{2n} = \{\delta_1^*, \dots, \delta_{2n}^*\}$ with the desired properties. By the pigeonhole principle, at least n elements in D_{2n} have the same label, say *white*. The other case will follow by a symmetric argument. We remove all black labeled elements in D_{2n} , and let $\{\delta_{j_1}^*, \dots, \delta_{j_n}^*\}$ be the resulting set, and for simplicity, let $\delta_{j_r}^* = \delta(v_{j_r}, v_{j_r+1})$.

Now consider the vertices $v_{j_1}, v_{j_2}, \dots, v_{j_n}, v_{j_n+1} \in V$. By construction and by Property II, we have $v_{j_1} < v_{j_2} < \dots < v_{j_n} < v_{j_n+1}$ and $\delta(v_{j_1}, v_{j_2}) = \delta_{i_{j_1}}^*, \delta(v_{j_2}, v_{j_3}) = \delta_{i_{j_2}}^*, \dots, \delta(v_{j_n}, v_{j_n+1}) = \delta_{i_{j_n}}^*$. Therefore, we have a monotone sequence

$$\delta(v_{j_1}, v_{j_2}) > \delta(v_{j_2}, v_{j_3}) > \dots > \delta(v_{j_n}, v_{j_n+1}).$$

By Lemma 3.1, there is a bad triple in the set $\{\delta_{j_1}^*, \dots, \delta_{j_n}^*\}$ with respect to ϕ . By Property IV, χ does not color every 4-tuple in $V = \{v_1, \dots, v_m\}$ blue which is a contradiction. \square

4 A new lower bound for $r_4(5, n)$

Again we apply a variant to the Erdős-Hajnal stepping up lemma in order to establish a new lower bound for $r_4(5, n)$. We start by establishing the following simple lemma.

Lemma 4.1. *For $n \geq 5$, there is an absolute constant $c > 0$ such that the following holds. For $N = \lfloor n^{c \log n} \rfloor$, there is a red/blue coloring ϕ on the pairs of $\{0, 1, \dots, N-1\}$ such that*

1. *there is no monochromatic red copy of $K_{\lfloor \log n \rfloor}$,*
2. *there are no two disjoint n -sets $A, B \subset \{0, 1, \dots, N-1\}$, such that $\phi(a, b) = \text{blue}$ for every $a \in A$ and $b \in B$ (i.e. no blue $K_{n,n}$).*
3. *there is no n -set $A \subset \{0, 1, \dots, N-1\}$ such that every triple $a_i, a_j, a_k \in A$, where $a_i < a_j < a_k$, avoids the pattern $\phi(a_i, a_j) = \phi(a_j, a_k) = \text{blue}$ and $\phi(a_i, a_k) = \text{red}$.*

Proof. Set $p = \log^2 n / 2n$ and let c be a sufficiently small constant that will be determined later. Consider the red/blue coloring ϕ on the pairs of $\{0, 1, \dots, N-1\}$, where $\phi(a, b) = \text{red}$ with probability p , and $\phi(a, b) = \text{blue}$ with probability $1 - p$. Then the expected number of monochromatic blue copies of the complete bipartite graph $K_{n,n}$ is at most

$$\binom{N}{n}^2 (1-p)^{n^2} < n^{cn \log n} e^{-pn^2} = e^{cn \log^2 n - n \log^2 n / 2} < 1/4$$

for c sufficiently small. Likewise, the expected number of monochromatic red copies of $K_{\lfloor \log n \rfloor}$ is at most

$$\binom{N}{\lfloor \log n \rfloor} p^{\frac{\log^2 n}{2}} < n^{c \log^2 n} \left(\frac{\log^2 n}{2n} \right)^{\frac{\log^2 n}{2}} < 1/4,$$

for c sufficiently small.

We call a triple $a_i, a_j, a_k \in \{0, 1, \dots, N-1\}$ *bad* if $a_i < a_j < a_k$ and $\phi(a_i, a_j) = \phi(a_j, a_k) = \text{blue}$ and $\phi(a_i, a_k) = \text{red}$. Otherwise, we call the triple (a_i, a_j, a_k) *good*. Now, let us estimate the expected number of sets $A \subset \{0, 1, \dots, N-1\}$ of size n such that every triple in A is good. For a given triple $a_i, a_j, a_k \in \{0, 1, \dots, N-1\}$, where $a_i < a_j < a_k$, the probability that (a_i, a_j, a_k) is good is $1 - p(1-p)^2$. Note that this is at most $1 - p/2$ as $p = \log^2 n / 2n$ and $n \geq 5$. Let $A = \{a_1, \dots, a_n\}$ be a set of n vertices in $\{0, 1, \dots, N-1\}$, where $a_1 < \dots < a_n$. Let S be a partial Steiner $(n, 3, 2)$ -system with vertex set A , that is, S is a 3-uniform hypergraph such that each 2-element set of vertices is contained in at most one edge in S . Moreover, S satisfies $|S| = 2c'n^2$ for some $c' > 0$. It is known that such a system exists. Then the probability that every triple in A is good is at most the probability that every edge in S is good. Since the edges in S are independent, that is no two edges have more than one vertex in common, the probability that every triple in A is good is at most $(1 - p/2)^{|S|} \leq e^{-c'n \log^2 n}$. Therefore, the expected number of sets of size n with every triple being good is at most

$$\binom{N}{n} e^{-c'n \log^2 n} < 1/4,$$

for an appropriate choice for c . By Markov's inequality and the union bound, we can conclude that there is a coloring ϕ with the desired properties. \square

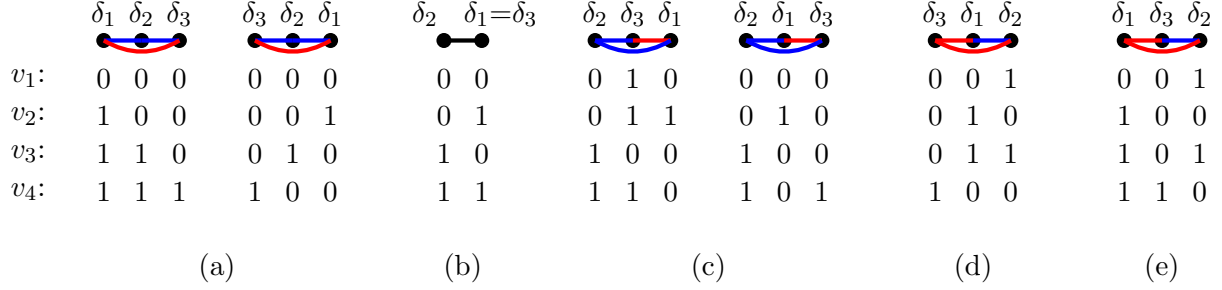


Figure 2: Examples of $v_1 < v_2 < v_3 < v_4$ and $\delta_1 = \delta(v_1, v_2), \delta_2 = \delta(v_2, v_3), \delta_3 = \delta(v_3, v_4)$, such that $\chi(v_1, v_2, v_3, v_4) = \text{red}$. For each case, v_i is represented in binary form with the left-most entry being the most significant bit.

For the reader's convenience, let us restate the result that we are about to prove.

Theorem 4.2. *For $n \geq 5$, there is an absolute constant $c > 0$ such that $r_4(5, n) > 2^{n^{c \log n}}$.*

Proof. Let $c > 0$ be the constant from Lemma 4.1, and set $A = \{0, 1, \dots, \lfloor n^{c \log n} \rfloor - 1\}$. Let ϕ be the red/blue coloring on the pairs of A with the properties described in Lemma 4.1. Set $N = 2^{\lfloor n^{c \log n} \rfloor}$ and let $V = \{0, 1, \dots, N - 1\}$. In what follows, we will use ϕ to define a red/blue coloring $\chi : \binom{V}{4} \rightarrow \{\text{red}, \text{blue}\}$ of the 4-tuples of V such that χ does not produce a monochromatic red $K_5^{(4)}$, and does not produce a monochromatic blue copy of $K_{2n^4}^{(4)}$. This would imply the desired lower bound for $r_4(5, n)$.

Just as in the previous section, for any $v \in V$, we write $v = \sum_{i=0}^{\lfloor n^{c \log n} \rfloor - 1} v(i)2^i$ with $v(i) \in \{0, 1\}$ for each i . For $u \neq v$, set $\delta(u, v) \in A$ denote the largest i for which $u(i) \neq v(i)$. Let $v_1, v_2, v_3, v_4 \in V$ such that $v_1 < v_2 < v_3 < v_4$ and set $\delta_i = \delta(v_i, v_{i+1})$. We define $\chi(v_1, v_2, v_3, v_4) = \text{red}$ if

- (a) $\delta_1, \delta_2, \delta_3$ is monotone and $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$, or
- (b) $\delta_1 < \delta_2 > \delta_3$ and $\delta_1 = \delta_3$, or
- (c) $\delta_1 < \delta_2 > \delta_3$, $\delta_1 \neq \delta_3$, and $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$, or
- (d) $\delta_1 > \delta_2 < \delta_3$, $\delta_1 < \delta_3$, and $\phi(\delta_1, \delta_3) = \phi(\delta_2, \delta_3) = \text{red}$ and $\phi(\delta_1, \delta_2) = \text{blue}$, or
- (e) $\delta_1 > \delta_2 < \delta_3$, $\delta_1 > \delta_3$, and $\phi(\delta_1, \delta_3) = \phi(\delta_1, \delta_2) = \text{red}$ and $\phi(\delta_2, \delta_3) = \text{blue}$.

See Figure 2 for small examples. Otherwise, $\chi(v_1, v_2, v_3, v_4) = \text{blue}$.

For sake of contradiction, suppose that the coloring χ produces a red $K_5^{(4)}$ on vertices $v_1 < \dots < v_5$, and let $\delta_i = \delta(v_i, v_{i+1})$, $1 \leq i \leq 4$. The proof now falls into the following cases, similar to the previous section.

Case 1. Suppose that $\delta_1, \dots, \delta_4$ forms a monotone sequence. If $\delta_1 > \dots > \delta_4$, then we have $\phi(\delta_1, \delta_3) = \text{red}$ since $\chi(v_1, v_2, v_3, v_4) = \text{red}$. However, this implies that $\chi(v_1, v_3, v_4, v_5) = \text{blue}$ since $\delta(v_1, v_3) = \delta_1$ by Property II, contradiction. A similar argument follows if $\delta_1 < \dots < \delta_4$.

Case 2. Suppose $\delta_1 > \delta_2 > \delta_3 < \delta_4$. By Property III, $\delta_4 \neq \delta_2, \delta_1$. Since $\delta_1 > \delta_2 > \delta_3$, this implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$. Now consider the following subcases for δ_4 .

Case 2.a. Suppose $\delta_4 > \delta_1$. By Property II, $\delta(v_2, v_4) = \delta_2$. Since $\chi(v_1, v_2, v_4, v_5) = \text{red}$, this implies that $\phi(\delta_4, \delta_1) = \phi(\delta_4, \delta_2) = \text{red}$. However, since $\delta_1 = \delta(v_1, v_3)$, this implies $\chi(v_1, v_3, v_4, v_5) = \text{blue}$, contradiction.

Case 2.b. Suppose $\delta_2 < \delta_4 < \delta_1$. Since $\chi(v_2, v_3, v_4, v_5) = \text{red}$, we have $\phi(\delta_4, \delta_2) = \phi(\delta_4, \delta_3) = \text{red}$. However, this implies that $\chi(v_1, v_2, v_4, v_5) = \text{blue}$ since $\delta(v_2, v_4) = \delta_2$, contradiction.

Case 2.c. Suppose $\delta_3 < \delta_4 < \delta_2$. Then this would imply $\chi(v_2, v_3, v_4, v_5) = \text{blue}$, contradiction.

Case 3. Suppose $\delta_1 < \delta_2 < \delta_3 > \delta_4$. This implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$. Hence we have $\delta_4 \neq \delta_1, \delta_2$. Since $\delta(v_2, v_4) = \delta_3$, we have $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction.

Case 4. Suppose $\delta_1 < \delta_2 > \delta_3 > \delta_4$. This implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) = \text{blue}$ and $\phi(\delta_2, \delta_4) = \text{red}$. Hence we have $\delta_1 \neq \delta_3, \delta_4$. Since $\delta(v_2, v_4) = \delta_2$, we have $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction.

Case 5. Suppose $\delta_1 > \delta_2 < \delta_3 < \delta_4$. Note that by Property III, $\delta_1 \neq \delta_3, \delta_4$. Since $\delta_2, \delta_3, \delta_4$ forms a monotone sequence, this implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) = \text{blue}$ and $\phi(\delta_2, \delta_4) = \text{red}$. Now we consider the following subcases for δ_1 .

Case 5.a. Suppose $\delta_2 < \delta_1 < \delta_3$. Then we have $\chi(v_1, v_2, v_3, v_4) = \text{blue}$ which is a contradiction.

Case 5.b. Suppose $\delta_3 < \delta_1 < \delta_4$. Then we have $\phi(\delta_1, \delta_3) = \phi(\delta_1, \delta_2) = \text{red}$. Notice that $\delta(v_2, v_4) = \delta_3$ by Property II. Therefore $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction.

Case 5.c. Suppose $\delta_1 > \delta_4$. Then we have $\phi(\delta_1, \delta_3) = \phi(\delta_1, \delta_2) = \text{red}$. By Property II, $\delta(v_3, v_5) = \delta_4$ which implies $\chi(v_1, v_2, v_3, v_5) = \text{blue}$, contradiction.

Case 6. Suppose $\delta_1 < \delta_2 > \delta_3 < \delta_4$. Then $\chi(v_1, v_2, v_3, v_4) = \text{red}$ implies that $\phi(\delta_2, \delta_1) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$. Now if $\delta_2 < \delta_4$, $\chi(v_2, v_3, v_4, v_5) = \text{red}$ implies that $\phi(\delta_4, \delta_2) = \phi(\delta_4, \delta_3) = \text{red}$. By Property II, we have $\delta(v_2, v_4) = \delta_2$, and therefore $\delta_1 < \delta_2 < \delta_4$. However, this implies $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction. Now if $\delta_4 < \delta_2$, then $\chi(v_2, v_3, v_4, v_5) = \text{blue}$, which is again a contradiction.

Case 7. Suppose $\delta_1 > \delta_2 < \delta_3 > \delta_4$. Then $\chi(v_2, v_3, v_4, v_5) = \text{red}$ implies that $\phi(\delta_3, \delta_2) = \phi(\delta_3, \delta_4) = \text{blue}$ and $\phi(\delta_2, \delta_4) = \text{red}$. Now if $\delta_1 < \delta_3$, then $\chi(v_1, v_2, v_3, v_4) = \text{blue}$ which is a contradiction. Therefore we can assume that $\delta_1 > \delta_3$. Since $\chi(v_1, v_2, v_3, v_4) = \text{red}$ we have $\phi(\delta_1, \delta_3) = \text{red}$. By Property II, $\delta(v_1, v_3) = \delta_1$ and $\delta_1 > \delta_3 > \delta_4$. This implies that $\chi(v_1, v_3, v_4, v_5) = \text{blue}$ which is a contradiction.

Next we show that there is no blue $K_m^{(4)}$ in coloring χ , where $m = 2n^4$. We will prove this statement via the following claims.

Claim 4.3. *There do not exist vertices $w_1 < \dots < w_n$ in V such that $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) = \text{red}$ for every $i < j < k$.*

Proof. Suppose for contradiction that these vertices $w_1 < \dots < w_n$ exist. Let $\delta_i = \delta(w_i, w_{i+1})$ and set $\delta_{i_1} = \max_i \delta_i$. Let $W = \{w_i : i \leq i_1\}$ and $W' = \{w_i : i > i_1\}$. By the pigeonhole principle, either $|W| \geq n/2$ or $|W'| \geq n/2$. Assume without loss of generality that $|W| \geq n/2$ and set $W_1 = W$. Observe that by hypothesis and definition of δ_{i_1} , for every $w_i, w_j \in W_1$, with $i < j$, we have

$$\phi(\delta(w_i, w_j), \delta_{i_1}) = \phi(\delta(w_i, w_j), \delta(w_j, w_{i_1+1})) = \text{red}.$$

Note that we obtain the same conclusion if $|W'| \geq n/2$ and $W_1 = W'$ since

$$\phi(\delta_{i_1}, \delta(w_i, w_j)) = \phi(\delta(w_{i_1}, w_i), \delta(w_i, w_j)) = \text{red}.$$

Now define $\delta_{i_2} = \max_{i < i_1} \delta_i$ and repeat the argument above to obtain W_2 with $|W_2| \geq n/4$ such that $\phi(\delta(w_i, w_j), \delta_{i_2}) = \text{red}$ for every $w_i, w_j \in W_2$, with $i < j$. Continuing in this way, we obtain $\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_m}$ for $m = \lfloor \log n \rfloor$, such that ϕ colors every pair in the set $\{\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_m}\}$ red. This contradicts Lemma 4.1, and the statement follows. \square

Claim 4.4. *There do not exist vertices $w_1 < \dots < w_{n^2}$ in V such that every 4-tuple among them is blue under χ and for every $i < j < k$ with $\delta(w_i, w_j) > \delta(w_j, w_k)$ we have $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) = \text{red}$.*

Proof. Suppose for contradiction that these vertices $w_1 < \dots < w_{n^2}$ exist. Let $\delta_i = \delta(w_i, w_{i+1})$ and set $\delta_{i_1} = \max_i \delta_i$. Let $W = \{w_i : i \leq i_1\}$ and $W' = \{w_i : i > i_1\}$. Let us first suppose that $|W'| \geq n$. Pick $w_i, w_j, w_k \in W'$ with $i < j < k$. If $\delta(w_i, w_j) > \delta(w_j, w_k)$, then $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) = \text{red}$ by assumption. If $\delta(w_i, w_j) < \delta(w_j, w_k)$, then consider the 4-tuple w_{i_1}, w_i, w_j, w_k . Since this 4-tuple is blue under χ , and both $\phi(\delta(w_{i_1}, w_i), \delta(w_i, w_j))$ and $\phi(\delta(w_{i_1}, w_i), \delta(w_j, w_k))$ are red, $\phi(\delta(w_i, w_j), \delta(w_j, w_k))$ must also be red. Now we may apply Claim 4.3 to W' to obtain a contradiction.

We may therefore assume that $|W'| < n$ and hence $|W| \geq n^2 - n \geq (n-1)^2$. We repeat the previous argument to W to obtain δ_{i_2} and then $\delta_{i_3}, \dots, \delta_{i_n}$, such that

$$\delta_{i_1} > \delta_{i_2} > \dots > \delta_{i_n} \quad \text{and} \quad i_1 > i_2 > \dots > i_n.$$

Now consider the set $S = \{w_{i_1+1}, w_{i_2+1}, \dots, w_{i_n+1}, w_{i_n}\}$, whose corresponding delta set is $A = \{\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_n}\}$. Then A is an n -set that has the properties of Lemma 4.1 part 3. This implies that there are $j < k < l$ such that $\phi(\delta_{i_j}, \delta_{i_k}) = \phi(\delta_{i_k}, \delta_{i_l}) = \text{blue}$ and $\phi(\delta_{i_j}, \delta_{i_l}) = \text{red}$. Consequently, $\chi(w_{i_j}, w_{i_k}, w_{i_l}, w_{i_l+1}) = \text{red}$, a contradiction. \square

By copying the proof above almost verbatim, we have the following.

Claim 4.5. *There do not exist vertices $w_1 < \dots < w_{n^2}$ in V such that every 4-tuple among them is blue under χ and for every $i < j < k$ with $\delta(w_i, w_j) < \delta(w_j, w_k)$ we have $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) = \text{red}$.*

Now we are ready to show that there is no blue $K_m^{(4)}$ in coloring χ , where $m = 2n^4$. For sake of contradiction, suppose we have vertices $v_1, \dots, v_m \in V$ such that $v_1 < \dots < v_m$, and χ colors every 4-tuple in the set $\{v_1, \dots, v_m\}$ blue. Let $\delta_i = \delta(v_i, v_{i+1})$ for $1 \leq i \leq m-1$. Notice that by Observation 3.4 we have $\delta_i \neq \delta_j$ for $1 \leq i < j < m$.

Let $\delta_1^* = \max\{\delta_1, \dots, \delta_m\}$, where $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$. Set

$$V_1 = \{v_1, v_2, \dots, v_{i_1}\} \quad \text{and} \quad V_2 = \{v_{i_1+1}, v_{i_1+1}, \dots, v_m\}.$$

Now we establish the following lemma.

Lemma 4.6. *We have either $|V_1| < n^3 = m/2n$ or $|V_2| < n^3 = m/2n$.*

Proof of Lemma 4.6. For sake of contradiction, suppose $|V_1|, |V_2| \geq n^3$. Recall that $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$, $V_1 = \{v_1, v_2, \dots, v_{i_1}\}$, $V_2 = \{v_{i_1+1}, v_{i_1+1}, \dots, v_m\}$, and set $A_1 = \{\delta_1, \dots, \delta_{i_1-1}\}$ and $A_2 = \{\delta_{i_1+1}, \dots, \delta_{m-1}\}$. For $i \in \{1, 2\}$, let us partition $A_i = A_i^r \cup A_i^b$ where

$$A_i^r = \{\delta_j \in A_i : \phi(\delta_1^*, \delta_j) = \text{red}\} \quad \text{and} \quad A_i^b = \{\delta_j \in A_i : \phi(\delta_1^*, \delta_j) = \text{blue}\}.$$

Let us first suppose that $|A_i^b| \geq n$ for $i = 1, 2$. Fix $\delta_{j_1} \in A_1^b$ and $\delta_{j_2} \in A_2^b$, and recall that $\delta_{j_1} = \delta(v_{j_1}, v_{j_1+1})$ and $\delta_{j_2} = \delta(v_{j_2}, v_{j_2+1})$. By Observation 3.4, $\delta_{j_1} \neq \delta_{j_2}$, and by Property II, we have $\delta(v_{j_1+1}, v_{j_2}) = \delta_1^*$. Since $\chi(v_{j_1}, v_{j_1+1}, v_{j_2}, v_{j_2+1}) = \text{blue}$, this implies that $\phi(\delta_{j_1}, \delta_{j_2}) = \text{blue}$. Consequently, we have a monochromatic blue copy of $K_{n,n}$ in A with respect to ϕ , which contradicts Lemma 4.1 part 2.

Therefore we have $|A_1^b| \leq n$ or $|A_2^b| \leq n$. Let us first suppose that $|A_1^b| \leq n$. Since $|A_1| \geq n^3$, by the pigeonhole principle, there is a subset $R \subset A_1^r$ such that $R = \{\delta_j, \delta_{j+1}, \dots, \delta_{j+n^2-2}\}$, whose corresponding vertices are $U = \{v_j, v_{j+1}, \dots, v_{j+n^2-1}\}$. For simplicity and without loss of generality, let us rename $U = \{u_1, \dots, u_{n^2}\}$ and $\delta_i = \delta(u_i, u_{i+1})$ for $1 \leq i \leq n^2$. Now notice that $\phi(\delta(u_i, u_j), \delta(u_j, u_k)) = \text{red}$ for every $i < j < k$ with $\delta(u_i, u_j) > \delta(u_j, u_k)$. Indeed, since $\delta(u_i, u_j), \delta(u_j, u_k) \in R$ we have $\phi(\delta_1^*, \delta(u_i, u_j)) = \phi(\delta_1^*, \delta(u_j, u_k)) = \text{red}$. Since $\chi(u_i, u_j, u_k, v_{i_1+1}) = \text{blue}$, this implies that we must have $\phi(\delta(u_i, u_j), \delta(u_j, u_k)) = \text{red}$ by definition of χ . However, by Claim 4.4 we obtain a contradiction.

In the case that $|A_2^b| \leq n$, a symmetric argument follows, where we apply Claim 4.5 instead of Claim 4.4 to obtain the contradiction. \square

Now we can finish the argument that χ does not color every 4-tuple in the set $\{v_1, \dots, v_m\}$ blue by copying the proof of Lemma 3.3. In particular, we will obtain vertices $v_{j_1} < \dots < v_{j_{n+1}} \in \{v_1, \dots, v_m\}$ such that $\delta(v_{j_1}, v_{j_2}), \delta(v_{j_2}, v_{j_3}), \dots, \delta(v_{j_n}, v_{j_{n+1}})$ forms a monotone sequence. By Property IV and Lemma 4.1, χ will color a 4-tuple in the set $\{v_{j_1}, \dots, v_{j_{n+1}}\}$ red.

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